Malaysian Journal of Mathematical Sciences 13(3): 447–464 (2019)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Recursive Smith-type Iterative Algorithm to Solve a Class of Periodic Lyapunov Equations Arising in Periodic Model Reduction

Hossain, M. S.

Department of Mathematics and Physics, North South University, Bangladesh

E-mail: mohammad.hossain@northsouth.edu

Received: 26 May 2018
Accepted: 29 August 2019

ABSTRACT

Periodic control systems are of interest in many engineering and mechanical research. Many important analysis of periodic control systems directly links to the periodic matrix equations, i.e., periodic Lyapunov equations or periodic Riccati equations. This paper represents the iterative methods for solving a class of periodic Lyapunov equation, known as periodic projected discrete-time Lyapunov equation. A remarkable contribution of these types of equations are seen in control problems with periodic setting, and also in dimension reduction of periodic systems in descriptor forms. We explore the Smith iterations for the iterative solution of the projected discrete-time algebraic Lyapunov equation and analyze the cyclic and repeated structure of the periodic matrices to ensure the recursive computation of the periodic solutions. We also introduce an algorithm for computing the low-rank approximations of those iterative solutions. Computational results are illustrated at the end to report the efficiency of the suggested methods.

Keywords: Periodic systems, periodic projected Lyapunov equations, reflexive inverses, Smith iterations.

1. Introduction

In the last few decades much attention has been given to the study of periodic control systems and the problems that arise in the context of periodic control problems. Such problems are, for example, the periodic feedback control, periodic model reduction, robust control, periodic stabilization problems, analyzing circuit models with periodic inputs, multirate data sampling control systems, orbital motion modelling of spacecrafts, periodic attitude control of helicopter rotors, etc., see e.g. Bittanti and Colaneri (2000), Kuo et al. (2004), Nakhla and Gad (2005), Varga (2007). The mathematical representations of these problems generally have huge memory and computational complexity. These complexities may generate from different resources, for example, structures of the models, discretization schemes used to linearize the partial differential equations that describe the models, and many more.

However, a huge number of applications addressing these problems demands the solution of the periodic matrix equations associated to the mathematical models. Periodic matrix equations such as periodic Sylvester equations which are closely related to the analysis and design of discrete-time periodic control systems have been widely studies in Hajarian (2018a,b,c) using the biconjugate residual method and the conjugate direction method. The periodic discrete-time Lypunov equation, an another well known matrix equation in the study of discrete-time periodic control systems and is mainly used in discrete-time periodic model reduction and periodic feedback control problems, has been studies very detail in Benner et al. (2011), Chu et al. (2007), Kressner (2003), Varga (1997). Our study is focused on the discrete-time linear periodic time-varying (LPTV) systems in their descriptor form.

In the study of model reduction for the periodic discrete-time descriptor system the system matrices are very large and sparse, and the system can be a multi-input-multi-output (MIMO) model. Therefore, the dense computation of the solution of the associated Lyapunov equation is computationally expensive and sometime unrealistic. As a result, huge attention has been devoted to find iterative solutions of large scale periodic Lyapunov equations for LPTV systems in the discrete-time setting Benner et al. (2014). In this paper, we describe one such iterative technique, known as Smith iterations, for the iterative solutions of the projected discrete-time algebraic Lyapunov equations and ensure the recursive computation of the periodic solutions.

We consider here the LPTV system in the discrete-time case having dimension $\mathbf{n} = (n_0, n_1, \ldots, n_{K-1})$ as

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbf{Z}, \tag{1}$$

where $E_k \in \mathrm{R}^{n_{k+1}\times n_{k+1}}$, $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{n_{k+1}\times m_k}$, $C_k \in \mathrm{R}^{p_k\times n_k}$ are the system matrices, $x_k \in \mathrm{R}^{n_k}$ is the state, $u_k \in \mathrm{R}^{m_k}$ is the control input, and $y_k \in \mathrm{R}^{p_k}$ is output. In (1) all system matrices are K-periodic, and $K \geq 1$. Also, $\sum_{k=0}^{K-1} n_k = \bar{\mathbf{n}}$. Here E_k can be singular for all k.

Many important analysis of (1) are accomplished by performing a causal and noncausal separation of system (1). Those analysis are connected with the periodic matrix equations, known as projected periodic discrete-time algebraic Lyapunov equations (PPDALEs) Chu et al. (2007)

$$\begin{aligned} A_k X_k A_k^T - E_k X_{k+1} E_k^T &= -P_l(k) B_k B_k^T P_l(k)^T, \\ X_k &= P_r(k) X_k P_r(k)^T, \end{aligned} \tag{2}$$

where X_k are the periodic reachability Gramians, $X_K = X_0$, and $P_l(k), P_r(k)$, for $k = 0, 1, \ldots, K-1$, are the left and right spectral projectors of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues Chu et al. (2007), Stykel (2008). In the study of control theory, we know (2) as causal reachability Lyapunov equation. This type of equation appears in many applications, for example, in periodic state feedback problems, in model reduction of periodic descriptor systems, robust control and stabilization of periodic dynamical systems Benner et al. (2011), Chu et al. (2007), Kuo et al. (2004). For the causal observability Gramians, we inherit a similar type equation.

The computation the periodic projectors is one of the crucial task in finding solutions of (2). Consider the pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is *periodic stable* (pd-stable) Benner et al. (2014), Chu et al. (2007). Then one can find the periodic canonical decomposed form Sreedhar and Van Dooren (1994), Van Dooren (1979) of $\{(E_k, A_k)\}_{k=0}^{K-1}$ for $k = 0, 1, \ldots, K-1$, as

$$U_k E_k V_{k+1} = \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & E_k^b \end{bmatrix}, \; U_k A_k V_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix}, \tag{3}$$

where U_k, V_k are nonsingular and periodic. One can then define the left and right spectral projectors $P_l(k)$ and $P_r(k)$ Benner et al. (2011), Chu et al. (2007), Stykel (2008) defined as

$$P_l(k) = U_k^{-1} \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & 0 \end{bmatrix} U_k, \; P_r(k) = V_k \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & 0 \end{bmatrix} V_k^{-1}, \tag{4}$$

onto the deflating subspaces of $\{(E_k, A_k)\}_{k=0}^{K-1}$, for $k = 0, 1, \ldots, K-1$.

Numerical solution of (2) is very demanding in many applications where periodic control is sought. Many research have been devoted to this direction in last few decades. Direct method for the numerical solution of (2) has been considered in Benner et al. (2011), Chu et al. (2007) for time-varying system. This proposed method is based on the decomposition of $\{(E_k, A_k)\}_{k=0}^{K-1}$ using generalized periodic Schur form Kressner (2001), Varga (2004b) and solves the resulting periodic Sylvester and Lyapunov equations. Hence, it is expensive in terms of computations and one should avoid this for large scale problems.

Therefor, huge attention has been given to iterative solutions of (2). A very challenging task in the iterative computation for solutions of (2) is to restore the cyclic structure of the solution at each iteration step Benner et al. (2014). This structure preservation may demand the inversion the system matrices. For descriptor systems where the matrices E_k can be singular, direct inversion of matrices may not be possible. In this paper, we introduce the reflexive inverses of periodic matrix pairs that are defined via the canonical decomposition of the periodic matrix pairs. The left and right deflating projectors are used for this computation. Those periodic inverses are then used for solutions of (3) using Smith iteration Gugercin et al. (2003), Penzl (2000). The main idea is to reformulate the Smith method using those periodic inverses and then solve the periodic projected Lyapunov equation. It is worth mentioning that the generalized ADI method and the Smith method have been exploited in Benner et al. (2014) to find the approximate solutions of (3). But, the methods discussed in Benner et al. (2014) fail to preserve the demanding block diagonal structure of the solution during the iteration process.

2. Analysis of Numerical Solutions of PPDALEs

Direct solution of (2) using the corresponding lifted forms of (1) and (2) has been proposed in Benner et al. (2011). Using the lifted representation, one can show that the periodic Lyapunov equations (2) is equivalent to the following projected lifted discrete-time algebraic Lyapunov equation (PLDALE)

$$\mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{E}\mathcal{X}\mathcal{E}^T = -\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r\mathcal{X}\mathcal{P}_r^T, \tag{5}$$

where

$$\mathcal{E} = \mathrm{diag}(E_0, E_1, \ldots, E_{K-1}),\ \mathcal{B} = \mathrm{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathcal{A} = \begin{pmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{pmatrix}, \tag{6}$$

and

$$\begin{aligned} \mathcal{X} &= \mathrm{diag}(X_1, \ldots, X_{K-1}, X_0), \\ \mathcal{P}_l &= \mathrm{diag}(P_l(0), P_l(1), \ldots, P_l(K-1)),\ \mathcal{Q}_l = I - \mathcal{P}_l, \\ \mathcal{P}_r &= \mathrm{diag}(P_r(1), \ldots, P_r(K-1), P_r(0)), \mathcal{Q}_r = I - \mathcal{P}_r. \end{aligned} \tag{7}$$

In general, the direct method is not a wise choice for large-scale problems due to their computational complexity ($\mathcal{O}(Kn_{max}^3)$, where n_{max}=max(n_k)) and extensive use of memory. As a result, research has been developed to iterative solutions of (2). Iterative approach using the corresponding lifted structures of the PPDALEs, i.e., (5) has been considered in Benner et al. (2014), Hossain (2011). A brief discussion of implementing the generalized ADI and the Smith method can be found in Benner et al. (2014), Hossain (2011) for the solutions of (5).

However, the generalized ADI method proposed in Benner et al. (2014) fails to preserve the block diagonal structure at each iteration step of the approximate solution due to the singularity of the periodic matrices E_k, for $k = 0, 1, \ldots, K-1$. Therefore, the concept of generalized inverses of E_k has been raised. A class of generalized inverse, known as reflexive generalized inverse of periodic matrix pairs can resolve the issue by preserving the block diagonal structure in the iterative computation.

3. Reflexive inverses of periodic matrix pairs

For periodic systems with non-singular E_k, the inversion problem has exploited in Kono (1981), Perdon et al. (1992) to analyse the important properties of periodic systems. A generalization version of that inversion problem has been considered in Varga (2004b) using the lifted representation. For the generalized LTI case, the *reflexive generalized inverse* has been proposed in Stykel and Simoncini (2012) to find the inverses of singular system pencil. In Benner and Sokolov (2006), a similar concept of reflexive inverses has been proposed to compute the partial realization for descriptor systems. Similar inverses have also been proposed in Varga (2004a) with the name *(1,2)-inverse*. Brief description of these definitions can be found in Campbell and Meyer (2009). Analogous to Stykel and Simoncini (2012), we reformulate the *reflexive generalized inverses* for the periodic matrices E_k with respect to $P_l(k)$ and $P_r(k+1)$ as

$$\bar{E}_k = V_{k+1} \begin{bmatrix} I_{n^f_{k+1}} & 0 \\ 0 & 0 \end{bmatrix} U_k, \tag{8}$$

for $k = 0, 1, \ldots, K-1$. For nonsingular A_k, there is no need to construct such inversion. Because, the exact inverse of A_k is equal to its reflexive generalized inverse for each $k = 0, 1, \ldots, K-1$. The main advantage of this approach is that the resulting system preserves the block sparsity of the lifted system matrices. Moreover, the reflexive generalized inverses satisfy

$$\bar{E}_k E_k \bar{E}_k = \bar{E}_k, \quad E_k \bar{E}_k = P_l(k), \quad \bar{E}_k E_k = P_r(k+1), \tag{9}$$

for $k = 0, 1, \ldots, K-1$.

4. Smith iterations for causal PLDALEs

The generalized Smith method for computing the solution of the causal PLDALEs can be exploited by multiplying the PLDALE (5) by $\bar{\mathcal{E}}$ and its inverse. Multiplying (5) from left and right by $\bar{\mathcal{E}}$, and $(\bar{\mathcal{E}})^T$, we get

$$\begin{aligned} \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T - \bar{\mathcal{E}} \mathcal{A} \mathcal{X} \mathcal{A}^T (\bar{\mathcal{E}})^T &= \bar{\mathcal{E}} \mathcal{P}_l \mathcal{B} \mathcal{B}^T \mathcal{P}_l^T (\bar{\mathcal{E}})^T, \\ \mathcal{X} &= \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \end{aligned} \tag{10}$$

where $\bar{\mathcal{E}}\mathcal{E} = \mathcal{P}_r$, and $\bar{\mathcal{E}} = \mathrm{diag}(\bar{E}_0, \bar{E}_1, \cdots, \bar{E}_{K-1})$. Equation (10) can be rewritten as

$$\mathcal{X} - (\bar{\mathcal{E}}\mathcal{A})\mathcal{X}(\bar{\mathcal{E}}\mathcal{A})^T = \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}\ (\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B})^T,\ \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \tag{11}$$

where $\mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T$, and $\mathcal{P}_r \bar{\mathcal{E}} = \bar{\mathcal{E}} \mathcal{P}_l$. Then (11) can be solved using the Smith method Penzl (2000), Stykel (2008) given by

$$\begin{aligned} \mathcal{X}_1 &= \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}\ (\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B})^T, \\ \mathcal{X}_\ell &= \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}\ (\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B})^T + (\bar{\mathcal{E}}\mathcal{A})\mathcal{X}_{\ell-1}(\bar{\mathcal{E}}\mathcal{A})^T. \end{aligned} \tag{12}$$

The unique solution $\mathcal{X}$ of (11) is approximated as

$$\mathcal{X}_i = \sum_{\ell=0}^{i-1} (\bar{\mathcal{E}}\mathcal{A})^\ell \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}\ \mathcal{B}^T \bar{\mathcal{E}}^T \mathcal{P}_r^T ((\bar{\mathcal{E}}\mathcal{A})^T)^\ell. \tag{13}$$

Hence, the Cholesky factor $\mathcal{R}_i$, defined as $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T$, is given by

$$\mathcal{R}_i = [\ \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B},\ (\bar{\mathcal{E}}\mathcal{A})\, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}, \ldots,\ (\bar{\mathcal{E}}\mathcal{A})^{i-1}\, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}\ ]. \tag{14}$$

Remark 4.1. At each iteration step i, (14) fails to generate the block diagonal structure as in (7) due to the presence of different block cyclic matrices at different iterative steps of (14). It is worth mentioning that $\mathcal{X} = \text{diag}(X_1, \ldots, X_{K-1}, X_0)$, and $X_k = R_k R_k^T$ for $k = 0, 1, \ldots, K-1$. Therefore, the main challenging task remains to compute the block diagonal Cholesky factor as $\mathcal{R}_i = \text{diag}(R^b_{1,i}, \ldots, R^b_{K-1,i}, R^b_{0,i})$ at each iteration step i of (14), where $R^b_{k,i}$ collects all the iterative counterparts of R_k at the i-th steps for $k = 0, 1, \ldots, K-1$, and $X_k = R_k R_k^T \approx R^b_{k,i}(R^b_{k,i})^T$. .

Remark 4.2. For Observability type periodic Lyapunov equations one gets a similar type lifted Lyapunov equation as in (10). This Lifted equation also fails to preserve the block diagonal structure in its iterative approximation. .

4.1 Structure Preserving Cyclic Computation of PLDALEs

Preserving the appropriate structure of the solution in the iterative computations is one of the most challenging tasks in periodic computation. The structure problem in (14) can be resolved by introducing a new permutation matrix in each iteration step i of (14). The permutation matrices introduced at different iteration steps also have nice cyclic structures. First, consider the trivial case

$$\Pi = \begin{bmatrix} S_{m_0} & 0 & \cdots & 0 & 0 \\ \vdots & S_{m_1} & 0 & & 0 \\ & & \ddots & & \vdots \\ 0 & & & S_{m_{K-2}} & 0 \\ 0 & 0 & \cdots & & S_{m_{K-1}} \end{bmatrix}, \tag{15}$$

where S_{m_i} denotes a square identity matrix of size m_i and m_i is the number of columns of B_i. The permutation matrix has the following two properties:

- $\sigma^i \Pi$ indicates a forward i block-column shift of Π.
- $\sigma^K \Pi = \sigma^0 \Pi = \Pi$.

We can consider few sample calculations like the following. Assume $K = 3$ $(k = 0, 1, 2)$, Then for $i = 1, \sigma^0 \Pi = \Pi$ is given by

$$\Pi = \begin{bmatrix} S_{m_0} & 0 & 0 \\ 0 & S_{m_1} & 0 \\ 0 & 0 & S_{m_2} \end{bmatrix}. \tag{16}$$

For $i = 2$, we compute

$$\sigma^1\Pi = \sigma\Pi = \sigma(\Pi) = \begin{bmatrix} 0 & S_{m_0} & 0 \\ 0 & 0 & S_{m_1} \\ S_{m_2} & 0 & 0 \end{bmatrix}, \tag{17}$$

which is a forward shift of the last block-column of Π in (16). Clearly for $i = 3$, we have

$$\sigma^2\Pi = \sigma(\sigma\Pi) = \sigma(\sigma(\Pi)) = \begin{bmatrix} 0 & 0 & S_{m_0} \\ S_{m_1} & 0 & 0 \\ 0 & S_{m_2} & 0 \end{bmatrix}, \tag{18}$$

which is simply a forward shift of the last block-column of $\sigma\Pi$. For iterative solutions for periodic Laypunov equations a similar types permutations are also seen in Kressner (2003).

Using the above permutation strategies, (13) takes the new form

$$\mathcal{X}_i = \sum_{\ell=0}^{i-1} (\bar{\mathcal{E}}\mathcal{A})^{\ell} \mathcal{P}_r \bar{\mathcal{E}}\mathcal{B}(\sigma^{\ell}\Pi)\,(\sigma^{\ell}\Pi)^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}\bar{\mathcal{E}}^{\mathrm{T}}\mathcal{P}_r^{\mathrm{T}}((\bar{\mathcal{E}}\mathcal{A})^{\mathrm{T}})^{\ell}.$$

Hence, the Cholesky factor $\mathcal{R}_i$ has the form

$$\mathcal{R}_i = [\, \mathcal{P}_r\bar{\mathcal{E}}\mathcal{B}\Pi,\ (\bar{\mathcal{E}}\mathcal{A})\,\mathcal{P}_r\,\bar{\mathcal{E}}\mathcal{B}(\sigma\Pi), \ldots,\ (\bar{\mathcal{E}}\mathcal{A})^{i-1}\,\mathcal{P}_r\,\bar{\mathcal{E}}\mathcal{B}(\sigma^{i-1}\Pi)\,]. \tag{19}$$

In (19) we have $\mathcal{P}_r\bar{\mathcal{E}}\mathcal{B}\Pi = \mathcal{P}_r\bar{\mathcal{E}}\mathcal{B}$, and $\mathcal{R}_i = \mathcal{P}_r\mathcal{R}_i$ is satisfied at each iteration step i. One can stop the approximations in (19) by meeting the criteria that the *normalized residual norm* defined as

$$\eta(\mathcal{R}_i) = \frac{\|\mathcal{A}\mathcal{R}_i\mathcal{R}_i^T\mathcal{A}^T - \mathcal{E}\mathcal{R}_i\mathcal{R}_i^T\mathcal{E}^T + \mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F}{\|\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F} \tag{20}$$

satisfies the condition $\eta(\mathcal{R}_i) < tol$. Here we define the tolerance level, tol a priori.

It is worth mentioning that each block in $\mathcal{R}_i$ has block diagonal structure, but $\mathcal{R}_i$ as a whole is not block diagonal. In fact, the diagonal blocks of (19) enables us to construct the algorithm for the periodic computation of the Cholesky factors directly from (14) for $k \in [0, K-1]$. With algebraic manipulation over (19), we find that the periodic matrices E_k, A_k, and B_k appear in a cyclic manner in periodic computations $R_{k,i}$ at the iteration step i for different k. Identifying and investigating these cyclic relations, we develop the algorithm for computing the periodic Cholesky factors $R_{k,i}$, $k = 0, 1, \ldots, K-1$,

$i = 1, 2, \ldots$, from the periodic coefficient matrices, directly. We represent few steps of those computations below: For $i = 1$, we get

$$\begin{aligned} R_{0,1} &= P_r(K)\bar{E}_{K-1}B_{K-1} = P_r(0)\bar{E}_{K-1}B_{K-1}, \\ R_{1,1} &= P_r(1)\bar{E}_0B_0, \qquad \% \ \bar{E}_K = \bar{E}_0,\ B_K = B_0 \\ &\vdots \\ R_{K-1,1} &= P_r(K-1)\bar{E}_{K-2}B_{K-2}. \end{aligned}$$

For $i = 2$, we get

$$\begin{aligned} R_{0,2} &= \bar{E}_{K-1}A_{K-1}P_r(K-1)\bar{E}_{K-2}B_{K-2}, \\ R_{1,2} &= \bar{E}_0A_0P_r(0)\bar{E}_{K-1}B_{K-1}, \\ &\vdots \\ R_{K-1,2} &= \bar{E}_{K-2}A_{K-2}P_r(K-2)\bar{E}_{K-3}B_{K-3}. \end{aligned}$$

For $i = 3$, we get

$$\begin{aligned} R_{0,3} &= \bar{E}_{K-1}A_{K-1}\bar{E}_{K-2}A_{K-2}P_r(K-2)\bar{E}_{K-3}B_{K-3}, \\ R_{1,3} &= \bar{E}_0A_0\bar{E}_{K-1}A_{K-1}P_r(K-1)\bar{E}_{K-2}B_{K-2}, \\ &\vdots \\ R_{K-1,3} &= \bar{E}_{K-2}A_{K-2}\bar{E}_{K-3}A_{K-3}P_r(K-3)\bar{E}_{K-4}B_{K-4}, \end{aligned}$$

and so on. Clearly, $R_{0,1} = R_{K,1}, R_{0,2} = R_{K,2}$ and $R_{0,3} = R_{K,3}$ in the above computation. We summarized the computation in Algorithm 1. Note that the periodicity of all the involved coefficient matrices including the projectors are considered in the above computation. That means, $P_r(K) = P_r(0)$, $P_r(K-1) = P_r(-1)$, $\bar{E}_K = \bar{E}_0$, $\bar{E}_{-1} = \bar{E}_{K-1}$, and the similar for others. Obviously, $X_K = X_0 = R_K R_K^T = R_0 R_0^T$. In Algorithm 1, $R_{k,i}$ means the approximated periodic Cholesky factor R_k computed at the i-th iteration steps. Finally, $R_{k,i}^b$ collects all these computed components for a specific k, where $k = 0, 1, \ldots, K-1$. For example, we compute $R_{0,i}^b = [R_{0,1}, R_{0,2}, \ldots, R_{0,i}]$ for $k = 0$. A similar construction is followed for the other values of k.

We can set a stopping criteria for the above iterative computation by setting the *normalized periodic residual norm* defined as

$$\eta(R_k) = \frac{\|A_k R_k R_k^T A_k^T - E_k R_{k+1} R_{k+1}^T E_k^T + P_l(k) B_k B_k^T P_l(k)^T\|_F}{\|P_l(k) B_k B_k^T P_l(k)^T\|_F}, \qquad (21)$$

which satisfies the condition $\eta(R_k) < tol_c$ for $k \in [0, K-1]$, where tol_c is the tolerance defined by the user.

Algorithm 1 Computing Periodic Cholesky factors using Smith method.

Input: $(\bar{E}_k, A_k, B_k)$, $P_r(k)$ for $k = 0, \dots, K-1$.
Output: Low-rank R_k, such that $X_k = R_k R_k^T$.

for $k = 0 : K-1$ **do**
  $R_{k,1} = P_r(k)\bar{E}_{k-1}B_{k-1}$ % $\bar{E}_{-1} = \bar{E}_{K-1}, B_{-1} = B_{K-1}$
  $P_{k,1} = I_{n_k}$ % initialization of a cyclic matrix
  $R^b_{k,1} = R_{k,1}$
end for
for $i = 2, 3, \dots$ **do**
  $m = \mathrm{mod}(i-2, K)$
  for $k = 0 : K-1$ **do**
    $P_{k,i} = P_{k,i-1}\bar{E}_{k-m-1}\,A_{k-m-1}$
    $R_{k,i} = P_{k,i}\,P_r(k-m-1)\,\bar{E}_{k-m-2}\,B_{k-m-2}$
    $R^b_{k,i} = [R^b_{k,i-1} \quad R_{k,i}]$
    $R_k = \mathrm{RRQR}(R^b_{k,i},\ \tau_k)$
  end for
end for

Remark 4.3. We compute the causal periodic observability Gramians of system (1) using a similar type computation strategy. In that case the permutation matrix changes in every iteration step by a backward shift of the last block-column. The corresponding cyclic computations of the periodic Cholesky factors are performed analogously.

5. Results

Example 1:

To test our derived algorithm, we consider an artificial model which is discrete and of index-1 (Benner et al., 2014, Example 1). The problem is reconstructed from its original periodic descriptor model in (Chu et al., 2007, Example 1). The periodic descriptor system has $n_k = 404$, $m_k = 2$ and $p_k = 3$ and the periodicity is $K = 10$, i.e., $k = 0, 1, \dots, 9$. We compute $P_l(k)$ and $P_r(k)$ using the Kronecker-decomposition of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ proposed in Chu et al. (2007), Varga (1995). Each periodic matrix pairs of $\{(E_k, A_k)\}_{k=0}^{K-1}$, for $k = 0, 1, \dots, 9$, has $n_k^f = 400$ causal eigenvalues and $n_k^\infty = 4$ noncausal eigenvalues. The set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable.

The constructed lifted system has order $\mathbf{n} = 4040$. Since E_k are singular, we compute the $\bar{E}_k$ using relation (8).

We illustrate the data structure plotting the sparsity of the periodic pair for $k = 0$ in Figure 1. A similar type sparsity is also observed for other matrix pairs.

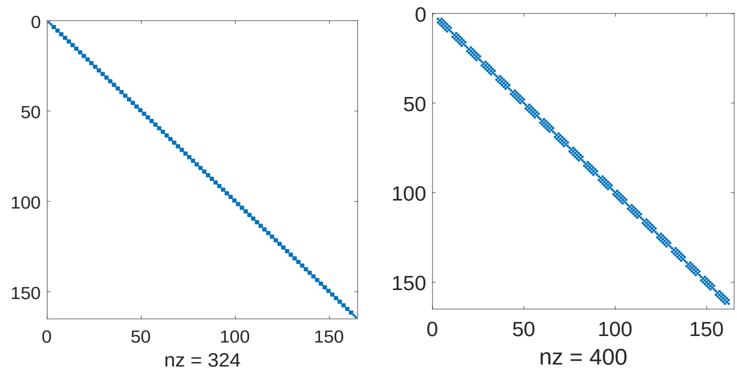

Figure 1: Sparsity patterns of A_0 (left) and E_0 (right).

The causal eigen-spectrum of the lifted system is shown in Figure 2, which shows that all the causal eigenvalues of the lifted system has magnitude less than 1. Note that the eigenvalues of the lifted system are also eigenvalues of the corresponding periodic system. This also represents the stability of the original periodic system.

We find the approximate solution of the causal lifted Lyapunov equations using Algorithm 1. Since, Algorithm 1 is the cyclic reformulation of the approximation stated in (19), we compute the normalized residual norms for the lifted Lyapunov equations, both for the reachability and observability types. We use relation (20) to compute these residual norms. To reduce the redundancy we apply the rank-revealing QR (RRQR)Golub and Loan (1996) decomposition at every fifth iteration step, since computing norms at each iteration steps is computationally expensive. We repeat the iteration process 30 times to reach the tolerance level close to 10^{-10}. Figure 3 shows how the residual norms decrease at the Smith iterations during the computation of the reachability type causal lifted Lyapunov equations.

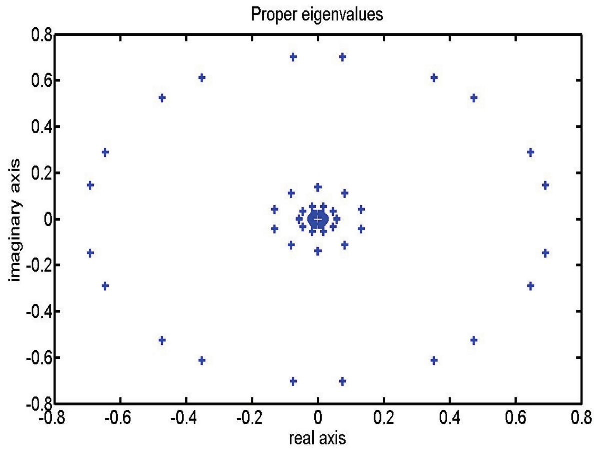

Figure 2: Finite eigen-spectrum of the pencil $\lambda\mathcal{E} - \mathcal{A}$.

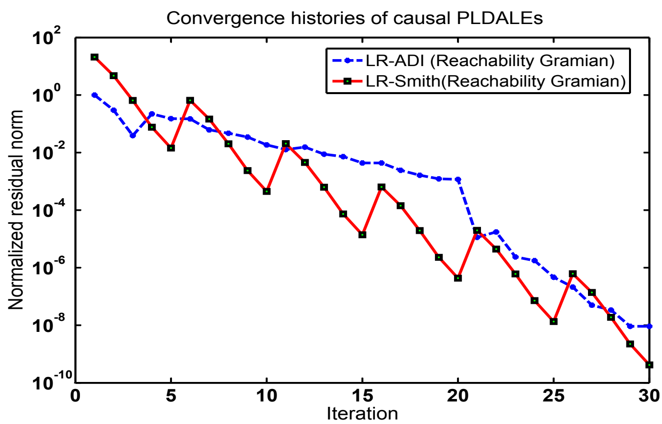

Figure 3: Normalized residual norms for the lifted projected Lyapunov equations for causal reachability type.

Figure 4 shows the similar results for the causal observability type lifted Lyapunov equations. We observe that within 30 iterations the residual reaches to a significant low numeric scale. We terminate the iteration procedure while the norms of the residual reaches the tolerance level $tol = 10^{-10}$. We emphasize that the computed results can be used further for model reduction of periodic system (1), and for periodic feedback control problem associate with system (1).

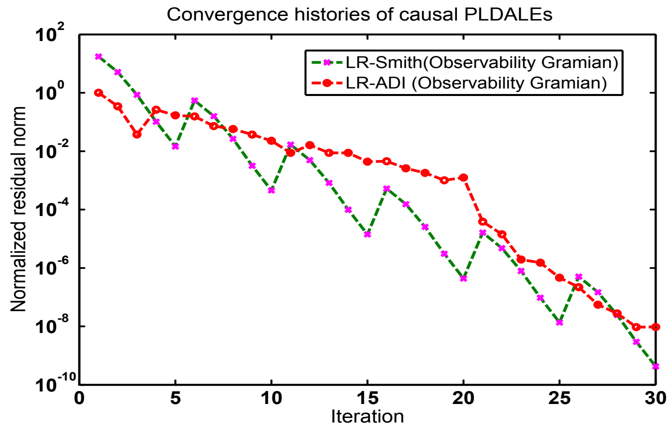

Figure 4: Normalized residual norms for the lifted projected Lyapunov equations for causal observability type.

Example 2:

We take the second example form Section 4.3 of Uddin (2011), where a spring-damper model is considered as an artificial model describing a piezo-mechanical system. The original model is a continuous-time model. We discretize the model to convert it into a discrete-time model and then impose periodic coefficients into the damping matrix. As a result, we get a discrete-time descriptor system of periodic dimensions $n_k = 1100$, $m_k = 2$ and $p_k = 3$ and the periodicity is $K = 10$. Details of the model formulation can be found in the Appendix of Benner and Hossain (2017).

For this reformulated periodic model we have $n_k^f = 1100$ and $n_k^\infty = 100$ for each pair of $\{(E_k, A_k)\}_{k=0}^{K-1}$, where $k = 0, 1, \ldots, 9$. The model is periodic stable and the constructed lifted system has order $\mathbf{n} = 11000$. Similar to Example 1, we compute the $\bar{E}_k$, for $k = 0, 1, \ldots, 9$, using relation (8).

We plot the sparsity of the periodic pair for $k = 0$ in Figure 5. A similar type sparsity is also observed for other matrix pairs.

Figure 6 shows how the residual norms decrease at the Smith iterations during the computation of the reachability type causal lifted Lyapunov equations. We consider 35 iterations to reach to the tolerance level close to $tol = 10^{-5}$.

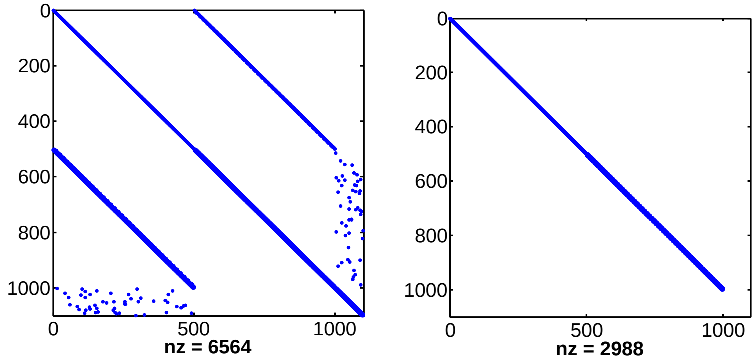

Figure 5: Sparsity patterns of A_0 (left) and E_0 (right).

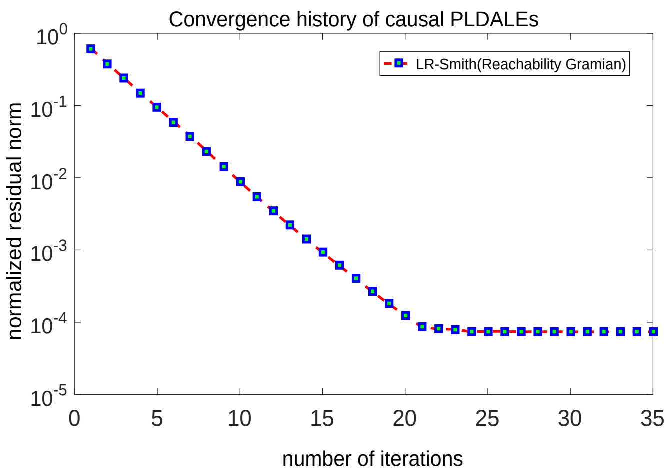

Figure 6: Normalized residual norms for the lifted projected Lyapunov equations for causal reachability type.

In Figure 7, we observe similar result for the causal observability type lifted Lyapunov equations.

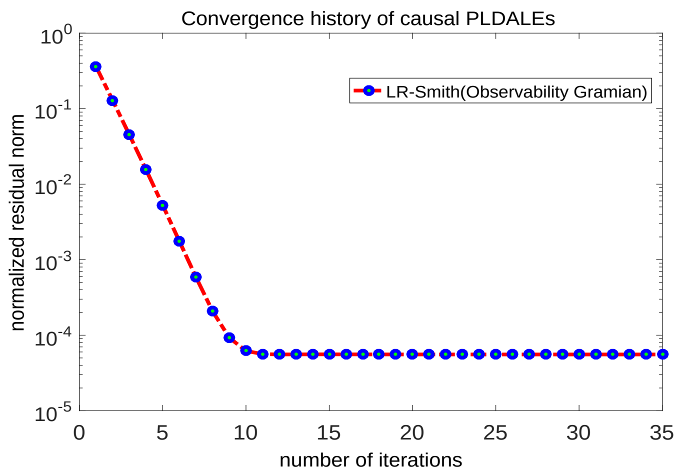

Figure 7: Normalized residual norms for the lifted projected Lyapunov equations for causal observability type.

6. Conclusion

In this paper we present details of the recursive computations of periodic Cholesky factors for the periodic Gramians which are the solutions of the generalized projected periodic discrete-time algebraic Lyapunov equations. The present work gives details of the iterative computations presented in Benner and Hossain (2017) to find model reduction of periodic descriptor system. The paper deals with the reflexive inverses of the periodic matrix pairs and exploits the low rank Smith method to compute the approximate Cholesky factors of the periodic Gramians. These periodic Gramians are the solutions of the periodic Lyapunov equations relating the concerned periodic system.

Acknowledgement

The author likes to thank North South University for the research grant NSU-RP-18-040 for FY 2018-2019. This research work is partially funded from this research grant.

References

Benner, P. and Hossain, M. (2017). Structure preserving iterative methods for periodic projected lyapunov equations and their application in model reduction of periodic descriptor systems. *Numerical Algorithms, Springer, DOI:https://doi.org/10.1007/s11075-017-0288-y*, pages 1–24.

Benner, P., Hossain, M.-S., and Stykel, T. (2011). *Model Reduction of Periodic Descriptor Systems Using Balanced Truncation*, volume 74 of *Lecture Notes in Electrical Engineering*. Springer-Verlag, Berlin, Heidelberg.

Benner, P., Hossain, M.-S., and Stykel, T. (2014). Low-rank iterative methods for periodic projected lyapunov equations and their application in model reduction of periodic descriptor systems. *Numerical Algorithms, Springer, DOI:10.1007/s11075-013-9816-6*.

Benner, P. and Sokolov, I. V. (2006). Partial realization of descriptor systems. *Systems Control Lett.*, 55(11):929–938.

Bittanti, S. and Colaneri, P. (2000). Invariant representations of discrete-time periodic systems. *Automatica*, 36:1777–1793.

Campbell, S. L. and Meyer, C. D. (2009). *Generalized Inverses of Linear Transformations*. SIAM, Philadelphia, PA.

Chu, E.-W., Fan, H.-Y., and Lin, W.-W. (2007). Projected generalized discrete-time periodic Lyapunov equations and balanced realization of periodic descriptor systems. *SIAM J. Matrix Anal. Appl.*, 29(3):982–1006.

Golub, G. and Loan, C. V. (1996). *Matrix Computations. 3rd ed.* The Johns Hopkins University Press, Baltimore, London.

Gugercin, S., Sorensen, D., and Antoulas, A. (2003). A modified low-rank Smith method for large-scale Lyapunov equations. *Numer. Algorithms*, 32(1):27–55.

Hajarian, M. (2018a). Finding solutions for periodic discrete-time generalized coupled sylvester matrix equations via the generalized bcr method. *Transactions of the Institute of Measurement and Control*, 40(2):647–656.

Hajarian, M. (2018b). Matrix form of biconjugate residual algorithm to solve the discrete-time periodic sylvester matrix equations. *Asian Journal of Control*, 20(1):49–56.

Hajarian, M. (2018c). Periodic conjugate direction algorithm for symmetric periodic solutions of general coupled periodic matrix equations. *Computers & Mathematics with Applications*, 75.

Hossain, M.-S. (2011). *Numerical Methods for Model Reduction of Time-Varying Descriptor Systems*. Ph.D. thesis, Chemnitz University Of Technology, Germany. Available from `http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-74776`.

Kono, M. (1981). Invertibility of linear time-varying discrete-time systems. *Systems Control Lett.*, 1(2):148–153.

Kressner, D. (2001). An efficient and reliable implementation of the periodic QZ algorithm. In *In IFAC Workshop on Periodic Control systems*.

Kressner, D. (2003). Large periodic Lyapunov equations: algorithms and applications. In *Proc. of ECC03, Cambridge, UK*.

Kuo, Y.-C., Lin, W.-W., and Xu, S.-F. (2004). Regularization of linear discrete-time periodic descriptor systems by derivative and proportional state feedback. *SIAM J. Matrix Anal. Appl.*, 25(4):1046–1073.

Nakhla, M. and Gad, E. (2005). Efficient model reduction of linear periodically time-varying systems via compressed transient system function. *IEEE Trans. Automat. Control*, 52(6):1188 – 1204.

Penzl, T. (2000). A cyclic low rank Smith method for large sparse Lyapunov equations. *SIAM J. Sci. Comput.*, 21:1401–1418.

Perdon, A. M., Conte, G., and Longhi, S. (1992). Invertibility and inversion of linear periodic systems. *Automatica*, 28:645–648.

Sreedhar, J. and Van Dooren, P. (1994). Periodic Schur form and some matrix equations. In Helmke, U., Mennicken, R., and Saurer, J., editors, *Systems and Networks: Mathematical Theory and Applications, Proc. Intl. Symposium MTNS '93 held in Regensburg, Germany, August 2–6, 1993*, pages 339–362. Akademie Verlag, Berlin, FRG.

Stykel, T. (2008). Low-rank iterative methods for projected generalized Lyapunov equations. *Electron. Trans. Numer. Anal.*, 30:187–202.

Stykel, T. and Simoncini, V. (2012). Krylov subspace methods for projected Lyapunov equations. *Appl. Numer. Math.*, 62:35–50.

Uddin, M, M. (2011). Model reduction for piezo-mechanical systems using balanced truncation. Master's thesis, Department of Mathematics, Stockholm University, Stockholm, Sweden.

Van Dooren, P. (1979). The computation of the Kronecker's canonical form of a singular pencil. *Linear Algebra Appl.*, 27:103–140.

Varga, A. (1995). Computation of Kronecker-like forms of a system pencil: Applications, algorithms and software. Technical Report R181–95, DLR–Oberpfaffenhofen, Institute for Robotics and System Dynamics, P.O.B. 1116, 82230 Wessling, Germany.

Varga, A. (1997). Periodic Lyapunov equations: some applications and new algorithms. *Internat. J. Control*, 67(1):69–87.

Varga, A. (2004a). Computation of generalized inverses of periodic systems. In *Proc. of the 2004 IEEE Conference on Decision and Control(CDC), Paradise Island, Bahamas.*

Varga, A. (2004b). Computation of Kronecker-like forms of periodic matrix pairs. In *Proc. of Mathematical Theory of Networks and Systems (MTNS 2004, Leuven, Belgium, July 5-9, 2004).*

Varga, A. (2007). An overview of recent developments in computational methods for periodic systems. In *(Proceedings of the Third IFAC Workshop on Periodic Control Systems (St. Petersburg, Russia, August 29-31, 2007).*